Reverse Fibonacci sequence and its description

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Abstract: This article describes a new sequence called *"Reverse Fibonacci sequence"*. The introduction of the study deals with the derivation of limits of a ratio of the two following numbers of the proposed sequence, which is equal to the number *j*. Further, the individual properties of this number are described. Another part of the article is devoted to the geometric interpretation of the obtained number *j*. With the help of the conclusions of the geometrical interpretation, the individual relations of the number *j* to the number *π*, the golden ratio *φ*, the Euler number e and spiral of Theodorus are derived. The conclusion of the article deals with the relationship of the proposed sequence to the Fibonacci sequence. The proposed sequence is characterized by the same features as the Fibonacci sequence for individual values of the digital root of members of its sequence. The values of digital roots of the proposed sequence are reversed to the Fibonacci sequence values.

Keywords: *Reverse Fibonacci sequence, Fibonacci sequence, Lucas sequence, digital root, number π, Euler number e, golden ratio φ, spiral of Theodorus, Vesica piscis*

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1. Introduction

The Fibonacci sequence and the golden ratio are well-known mathematical knowledge whose significance is gradually revealed. The use of the golden ratio and the Fibonacci sequence can be seen in nature almost everywhere, ranging from the pattern of sunflower seeds to the proportions of a nautilus. This ratio has even been applied in architecture, art and music, proving that people are subconsciously fascinated by the golden ratio. [1]

This article will describe a sequence that has much in common with the Fibonacci sequence, where these conclusions will be described in the following chapters. From the proposed sequence, further mathematical and geometric meanings will be derived.

2. Reverse Fibonacci sequence

As described above, a sequence was proposed in the following form, see equation (1). From the formula below, it is clear that the sequence members are only positive numbers, i.e. the members of the proposed sequence are from the natural numbers ***N***, where the individual members' digital root is just reversed to the members of the Fibonacci sequence. Obviously, other rows with similar patterns can be written, but these rows will not have unique properties, as described by the proposed reverse Fibonacci sequence. These properties will be further described in the following chapters.

The proposed sequence is a very similar to sequenceA057084, according to *OEIS*[1] [2], with the difference of the zero member. Sequence A057084 has for the member ***n=0*** value ***1***, while the

[1]*The On-Line Encyclopedia of Integer Sequences is a database of integer sequences freely available on the web. It contains over 300,000 sequences (by March 2018) that can be searched using both keywords and a section of the search sequence. Sequence of information is stored about each sequence, including links to other literature and websites related to a given sequence.*

proposed sequence has a value ***0*** for the member ***n=0.*** This fact is very important in terms of periodicity and relation to the Fibonacci sequence and will be described in the following chapters.

$$J(n) = \begin{cases} 0, & for\ n = 0; \\ 1, & for\ n = 1; \\ 8 \cdot (J(n-1) - J(n-2)) & n > 1. \end{cases} \tag{1}$$

To simplify calculations, this sequence can be written as follows, see equation (2).

$$J_{(n+2)} = 8 \cdot \left(J_{(n+1)} - J_n\right) \tag{2}$$

From the analysis of the relation (1) or (2) it is obvious that the limit of the two following numbers of the proposed sequence is equal to the number ***j***, see **Figure 1**.

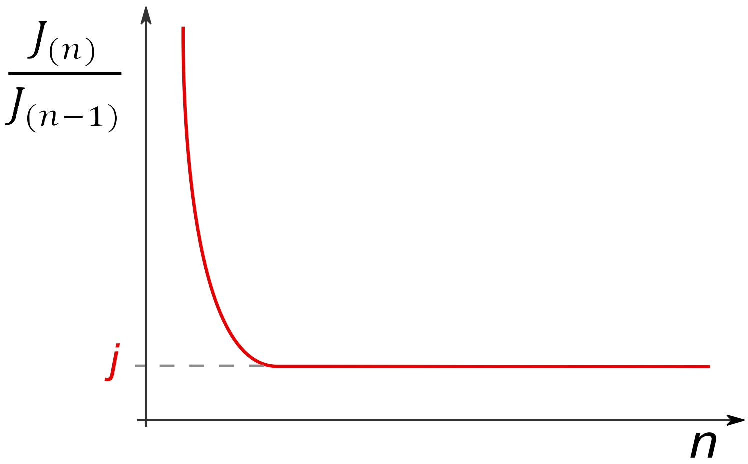

Figure 1. *Graphical expression of the ratio **j***

The ratio ***j*** can be written as follows, see equation (3) and put into the original relation for the proposed sequence (2).

$$\frac{J_{(n+2)}}{J_{(n+1)}} = j\ ;\ J_{(n+2)} = j \cdot J_{(n+1)}; \frac{J_{(n+1)}}{J_n} = j\ ; J_n = \frac{J_{(n+1)}}{j}\ . \tag{3}$$

By solving the resulting equation (4) a quadratic relation (5) is obtained.

$$\begin{gathered} j \cdot J_{(n+1)} = 8 \cdot \left(J_{(n+1)} - \frac{J_{(n+1)}}{j}\right) = 8 \cdot \left(\frac{j \cdot J_{(n+1)}}{j} - \frac{J_{(n+1)}}{j}\right) = 8 \cdot \left(\frac{J_{(n+1)} \cdot (j-1)}{j}\right) \\ j \cdot J_{(n+1)} = 8 \cdot \left(\frac{j \cdot J_{(n+1)}}{j} - \frac{J_{(n+1)}}{j}\right) \\ j \cdot J_{(n+1)} = 8 \cdot \left(\frac{J_{(n+1)} \cdot (j-1)}{j}\right) \\ \frac{j \cdot J_{(n+1)}}{8} \cdot \frac{j}{J_{(n+1)}} = (j-1) \\ \frac{j^2}{8} = (j-1) \end{gathered} \tag{4}$$

$$j^2 - 8j + 8 = 0 \tag{5}$$

By computing roots of the quadratic equation (5) we obtain the ratios j_1 and j_2 in the form, see eq. (6).

$$j_{1,2} = \frac{8 \pm \sqrt{32}}{2} = \frac{8 \pm 4\sqrt{2}}{2} = 4 \pm 2\sqrt{2},$$
$$j_1 \approx 6{,}82842712 \ldots$$
$$j_2 \approx 1{,}171572875 \ldots \tag{6}$$
$$\frac{1}{j_1} \approx 0{,}146446609 \ldots$$

The roots of the quadratic equation can be expressed, for example, by equations (7), (8), (9), (10), (11) and (12).

$$j_1 + j_2 = 8 \tag{7}$$

$$j_1 \cdot j_2 = 8 \tag{8}$$

$$j_1 \cdot j_2 = j_1 + j_2 \tag{9}$$

$$\frac{j_1}{j_2} = j_1 - 1 \tag{10}$$

$$\frac{{j_1}^2}{8} = j_1 - 1 \tag{11}$$

$$\frac{{j_2}^2}{8} = j_2 - 1 \tag{12}$$

From the above-mentioned relationships, it is clear that the **multiplication** or the **sum** of the individual roots of the ratio ***j*** number 8 is always obtained; see equations (7) and (8). This relation can be equated, see the relation (9). It is also apparent that the **ratio** of these roots is equal to the value of the difference of the **numerator** and the **number 1**, see equation (10). And last but not least, that the **eighth of a square area** of length ***j*** is equal to the distance of this **length with the difference of number 1**, see (11) and (12).

For clarification, the proposed sequence can be written using a recursive equation in form (13).

$$J(n) = \frac{\left(4 + 2\sqrt{2}\right)^n - \left(4 - 2\sqrt{2}\right)^n}{4\sqrt{2}} = \frac{\boldsymbol{{j_1}^n - {j_2}^n}}{\boldsymbol{4\sqrt{2}}} \tag{13}$$

From a more detailed examination of the proposed sequence and the properties of their roots, it follows that the proposed sequence is one of the Lucas sequences $U_n(P,Q)$, where ***P=Q=8.*** Such a sequence can be written in a form $\boldsymbol{U_n(8,8)}$,like the Fibonacci sequence in form $\boldsymbol{U_n(1,-1)}$.*[3]*

3. Geometric interpretation

For ease of illustration and simplification, only root j_1 is taken into account for further consideration. The geometric interpretation of the proposed sequence is based on the relation (11) as described above.

The left side of the equation (11) can be graphically represented by **Figure 2**.

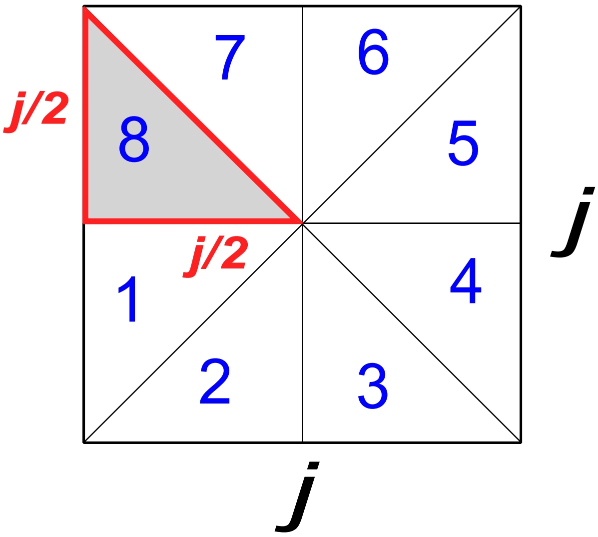

Figure 2. *Geometric representation of the left side of the equation (11)*

Figure 2 describes an isosceles triangle of side ***j***/2, where the area of this triangle is just equal to the left side of equation (11).

The right side of the equation (11) is represented by F**igure 3**.

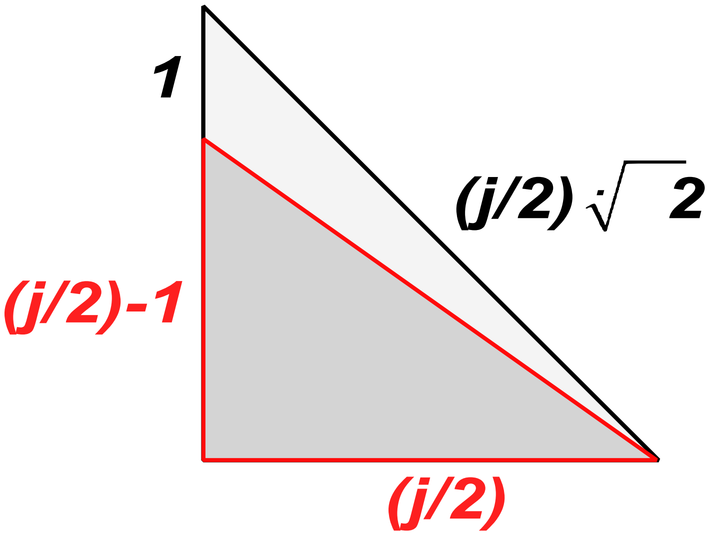

Figure 3. *Geometric representation of the right side of the equation (11)*

On **Figure 3** length ***(j-1)*** is depicted. By expressing this length, a rectangular triangle with the lengths of the legs ***j***/2 and ***(j***/2***)-1*** is constructed**.** On **Figure 3** is this triangle shown in **red** color.

Area of the original triangle ***S₁*** with leg length ***j*/2** is described by equation (14) and the area of the newly formed triangle ***S₂*** (on **Figure 3** drawn in **red** color) is described by equation (15).

$$S_1 = \frac{j^2}{8} \tag{14}$$

$$S_2 = \left(\frac{j}{2} - 1\right) \cdot \frac{j}{2} \cdot \frac{1}{2} = \frac{j^2}{8} - \frac{j}{4} = \frac{j \cdot (j-2)}{8} \tag{15}$$

If the areas (14) and (15) are put into ratio, the following relation is obtained, see equation (16).

$$\frac{S_1}{S2} = \frac{j^2}{8} \cdot \frac{8}{j \cdot (j-2)} = \frac{j}{j-2} \tag{16}$$

By assigning the root of the quadratic equation j_1 to equation (16) a relation (17) is obtained form:

$$\frac{S_1}{S2} = \frac{j_1}{j_1 - 2} = \sqrt{2}\,. \tag{17}$$

From the ratio of these areas when the root j_1 is substituted, a numerical value $\sqrt{2}$ is obtained. A similar solution, with the difference of triangles of length j, the relation (18) is obtained.

$$\frac{S_{12}}{S_{22}} = \frac{j}{j-1}\,. \tag{18}$$

By substituting the root of the quadratic equation j_1 to equation (18) the relation (19) is obtained in the following form:

$$\frac{S_{12}}{S_{22}} = \frac{j_1}{j_1 - 1} = j_2\,. \tag{19}$$

Thus, equation (18) describes that the ratio of the areas of the above triangles when the root j_1 is substituted is just equal to the root of the quadratic equation j_2.

The geometric representation of equation (17) is illustrated in **Figure 4**.

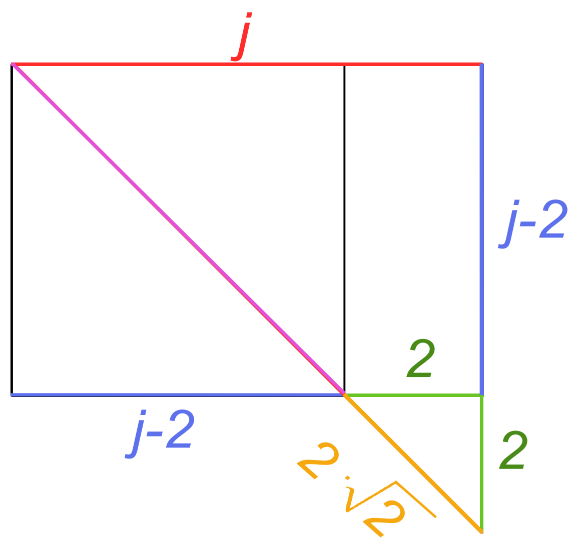

Figure 4. *Geometric representation of equation (17)*

Figure 4 describes the fact that if a rectangle is drawn with sides j and $(j-2)$, then the ratio of its individual lengths, when the root of a quadratic equation is substituted j_1, is equal to value $\sqrt{2}$ or $\frac{\sqrt{2}}{2}$.

This conclusion can be reached in another way, as shown in **Figure 4**. Again, it is the ratio of the individual areas, but in this case it is the ratio of the areas of a rectangle with sides ***j***, ***(j-2)*** and an isosceles triangle with the length of a base ***j.***

Area of a rectangle with legs ***j*** and ***(j-2)*** is represented by equation (20), and the area of an isosceles triangle with the length of base ***j*** by equation (21).

$$S_3 = j \cdot (j-2) \tag{20}$$

$$S_4 = \frac{j^2}{2} \tag{21}$$

The ratio of areas S_3 and S_4 is given by the following relation, see equation(22).

$$\frac{S_3}{S_4} = \frac{2j \cdot (j-2)}{j^2} = \frac{2 \cdot (j-2)}{j} \tag{22}$$

By substituting the root of the quadratic equation j_1 in equation (22), the relation (23) is obtained.

$$\frac{S_3}{S_4} = \frac{2 \cdot (j_1 - 2)}{j_1} = \sqrt{2} \tag{23}$$

Thus, it is obvious that a ratio of areas S_3 and S_4 is again equal to $\sqrt{2}$. For this reason, equations (17) and (23) can then be put into equation. This equation is described by the relation (24).

$$\frac{j}{j-2} = \frac{2 \cdot (j-2)}{j} \tag{24}$$

3.1 Relation of j to the number π

Using the equation (17), a rectangular isosceles triangle with leg lengths ***(j-2)*** and length of the hypotenuse ***j*** can be constructed. **Figure 5** illustrates the graphical representation of this triangle.

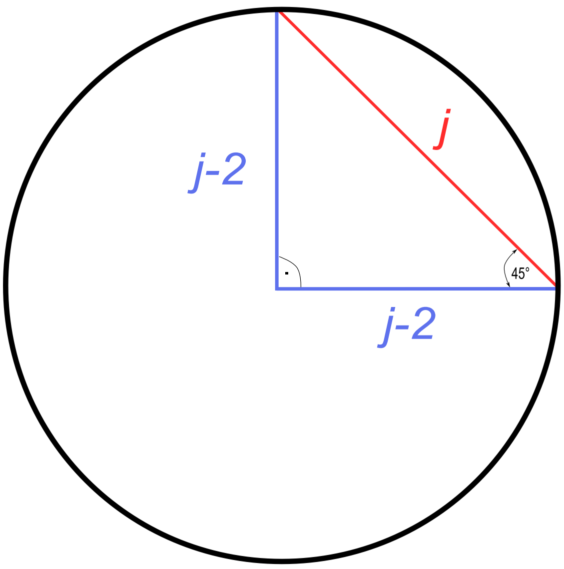

Figure 5. *Relation of j to the number* ***π***

As mentioned above, by the substitution of the root of the quadratic equation j_1 to equation (16), a relation whose value is equal to the number $\sqrt{2}$ is obtained. The inverted value of this relationship is then equal to $\frac{\sqrt{2}}{2}$.

Due to the isosceles triangle, it can be seen that the angle between the leg and the hypotenuse is always 45°, expressed in arc, as $\frac{\pi}{4}$.

Using the equation (17), the goniometric function can then be written as follows, see equation (25).

$$\sin\left(\frac{\pi}{4}\right) = \frac{j-2}{j} \tag{25}$$

By solving the relation (25) a relation between the ratio of **j** and the number **π** is obtained, see relation (26).

$$\pi = 4 \cdot arcsin\left(\frac{j-2}{j}\right) \tag{26}$$

The circumference of the circle of **Figure 5** can then be expressed only by means of the root of the quadratic equation j_1 in the following form (27). The area of this circle can be expressed similarly, see (28).

$$O_{jk} = 8(j-2) \cdot arcsin\left(\frac{j-2}{j}\right) \tag{27}$$

$$S_{jk} = 4 \cdot arcsin\left(\frac{j-2}{j}\right) \cdot (j-2)^2 \tag{28}$$

3.2 Relation of j to the golden ratio φ

The relation of j to the golden ratio φ is based on the basic formation of sacred geometry "*Vesica piscis*". This shape can be freely translated as "**bladder of a fish**" and is a pair of intersecting circles with centers on their circumferences. The center therefore has the length of a radius of one of them, see **Figure 6.** [4]

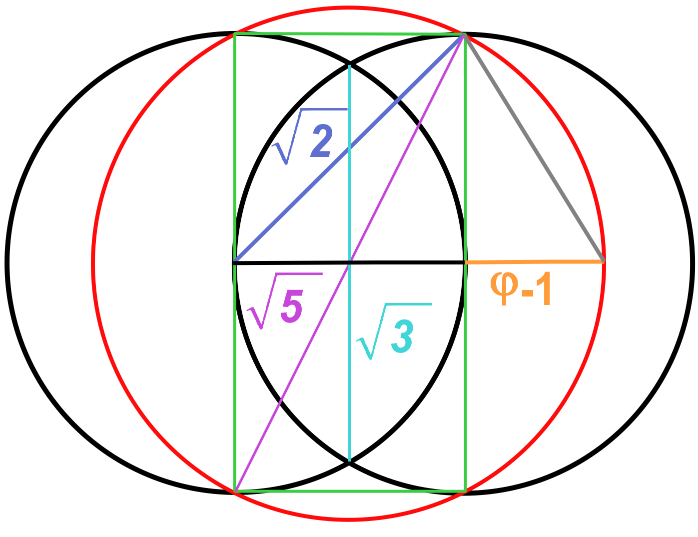

Figure 6. *Vesica piscis*

The ratio $\boldsymbol{j}$ can be expressed by means of the generalized triangle shown in **Figure 7**. The side $\boldsymbol{a}$ of the newly created triangle *(denoted by green on **Figure** 7)* is calculated using the Pythagorean Theorem and simple mathematical adjustments.

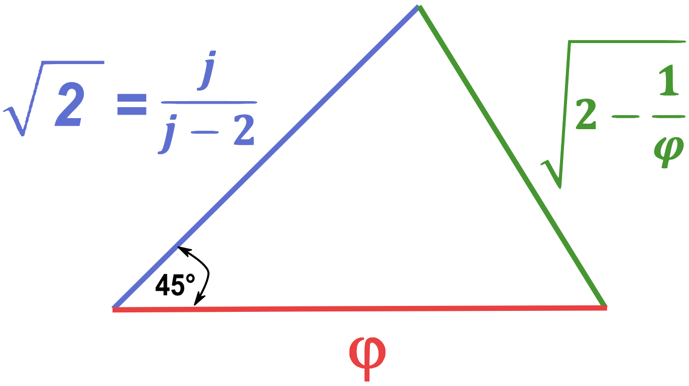

Figure 7. *Relation of j to the golden ratio* $\boldsymbol{\varphi}$

If all sides of a general triangle are known and the angle $\boldsymbol{\alpha} = \frac{\pi}{4}$, the ratio of $\boldsymbol{j}$ can be expressed using the **cosine sentence,** see derivation (29).

$$
\begin{gathered}
a^2 = b^2 + c^2 - 2bc \cdot cos\left(\frac{\pi}{4}\right) \\
2 - \frac{1}{\varphi} = 2 + \varphi^2 - 2\sqrt{2} \cdot \varphi \cdot \left(\frac{j-2}{j}\right) \\
\frac{\frac{1}{\varphi} + \varphi^2}{2\sqrt{2} \cdot \varphi} - 1 = -\frac{2}{j} \\
\boldsymbol{j} = \frac{\boldsymbol{4\sqrt{2} \cdot \varphi}}{\boldsymbol{2\sqrt{2} \cdot \varphi - \frac{1}{\varphi} - \varphi^2}}
\end{gathered}
\tag{29}
$$

The resulting relation (29) can also be written using simple mathematical operations as follows, see equation (30).

$$
\boldsymbol{j} = \frac{\boldsymbol{4\sqrt{2}}}{\boldsymbol{2\sqrt{2} - \varphi^2 + \varphi - 1}} \tag{30}
$$

Finally, by calculating the inverse function of the equation (30) we can express the relation of the golden ratio $\boldsymbol{\varphi}$ and ratio $\boldsymbol{j}$, see relation (31).

$$
\boldsymbol{\varphi_{1,2}} = \frac{\boldsymbol{1}}{\boldsymbol{2}} \pm \frac{\sqrt{\boldsymbol{8j\sqrt{2} - 3j - 16\sqrt{2}}}}{\boldsymbol{2\sqrt{j}}} \tag{31}
$$

3.3 Relation of j to Euler's number e

The relation of $\boldsymbol{j}$ to Euler's number $\boldsymbol{e}$ is based on the Euler formula. This formula determines the relation between goniometric functions and exponential function. This formula can be seen as a sentence of complex analysis, see equation (32). [5]

$$\boldsymbol{e^{i\alpha} = cos(\alpha) + i \cdot sin(\alpha)} \tag{32}$$

As mentioned above, by the substitution of the root of the quadratic equation j_1 to equation (16), a relation whose value is equal to the number $\sqrt{\mathbf{2}}$ is obtained. The inverted value of this relation is then equal to $\frac{\sqrt{2}}{2}$.

With the help of equation (17), the goniometric equations can then be written as follows, see equation (33) and **Figure 8**.

$$e^{\left(\frac{i\pi}{4}\right)} = cos\left(\frac{\pi}{4}\right) + i \cdot sin\left(\frac{\pi}{4}\right)$$
$$e^{\left(\frac{i\pi}{4}\right)} = \frac{j-2}{j} + i \cdot \left(\frac{j-2}{j}\right) \tag{33}$$

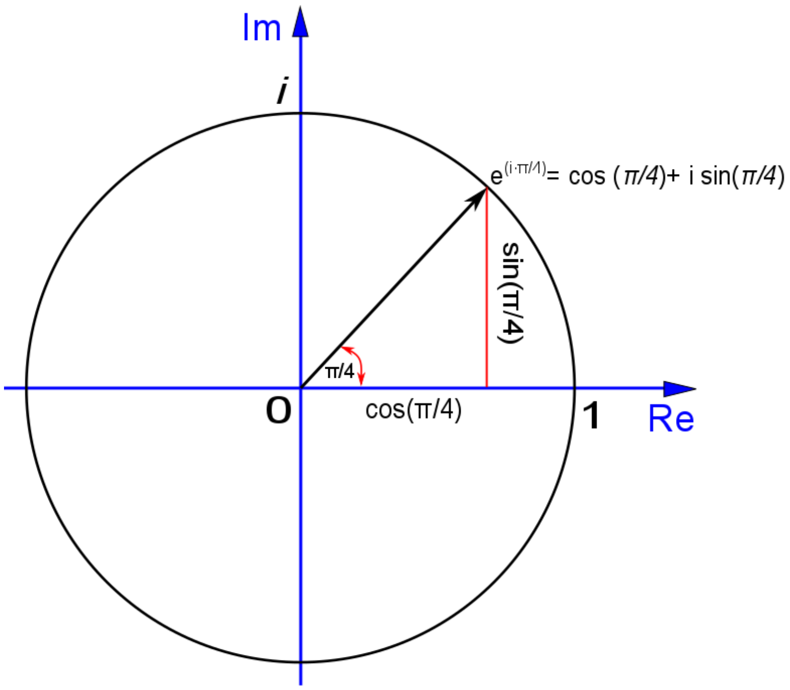

Figure 8. *Relation of **j** to Euler's number**e***

The resulting ratio relation of $\boldsymbol{j}$ to Euler's number $\boldsymbol{e}$ and its derivation from equation (33) is described by the following equation, see (34).

$$e^{\left(\frac{i\pi}{4}\right)} = \left(\frac{j-2}{j}\right) \cdot (1+i)$$
$$\frac{e^{\left(\frac{i\pi}{4}\right)}}{(1+i)} = \left(1 - \frac{2}{j}\right)$$
$$\frac{j}{2} = \frac{(1+i)}{1+i-e^{\left(\frac{i\pi}{4}\right)}} \tag{34}$$
$$\boldsymbol{j = \frac{2 \cdot (1+i)}{(1+i) - e^{\left(\frac{i\pi}{4}\right)}}}$$

3.4 Relation of j to the spiral of Theodorus

The construction of the spiral of Theodorus was created by Greek mathematician **Theodorus of Cyrene**. The spiral is formed by a sequence of rectangular triangles where the first of the triangles is isosceles rectangle whose legs are of length ***1***. The hypotenuse of the intruding triangle is the leg of the other, where the second leg is again equal to the value ***1***. The graphical representation of this spiral is shown in **Figure 9**. [6]

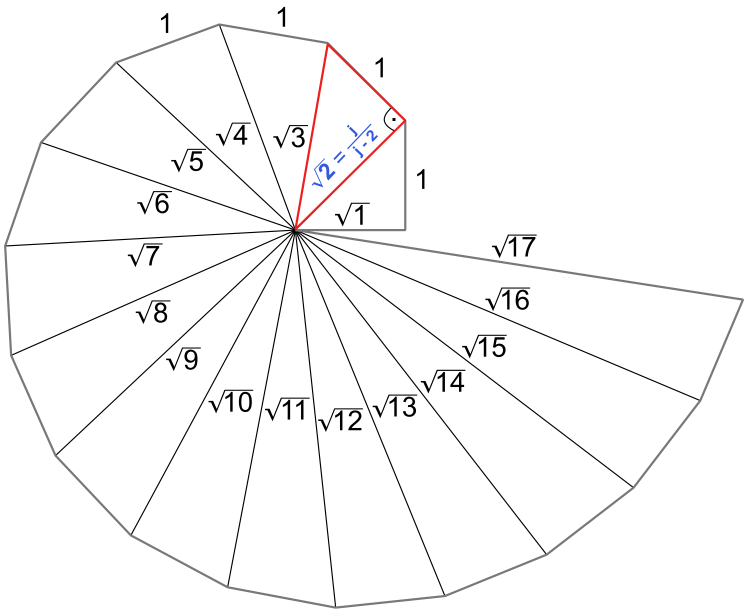

Figure 9. *Graphical representation of the spiral of Theodorus*

Ratio relation of j to the spiral of Theodorus is based on the **red** triangle shown in **Figure 9.** As in previous cases, the value $\sqrt{2}$ can be expressed using equation (17). By solving a simple equation, you can then write down any root of the natural number using the ratio ***j*.** This relation is described in equation (35).

$$\sqrt{n} = \sqrt{\frac{j^2}{(j-2)^2} + (n-2)}\,, n \in \; N \tag{35}$$

4. Relation of proposed sequence with Fibonacci sequence

In the introduction, the Fibonacci sequence and its relation to the proposed reverse sequence were mentioned. In the Fibonacci sequence, a recurrent pattern with ***n=24*** period is hidden. This pattern is obtained by the digital root[2] of individual members of the sequence, where the digits can be written using a congruent relation (36). [7]

$$dr(n) = 1 + [(n-1) mod\ 9] \tag{36}$$

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[2]*The digital root of a non-negative integer is the (single digit) value obtained by an iterative process of summing digits, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached.*

This period can be further subdivided into a **half period** of **n=12**, where the digital root of the members of Fibonacci sequence ***dr[F(n) + F(n+12)]*** result **9** is always obtained. The distribution of the digital roots of the individual members of the Fibonacci sequence is shown in the **Figure 10**.

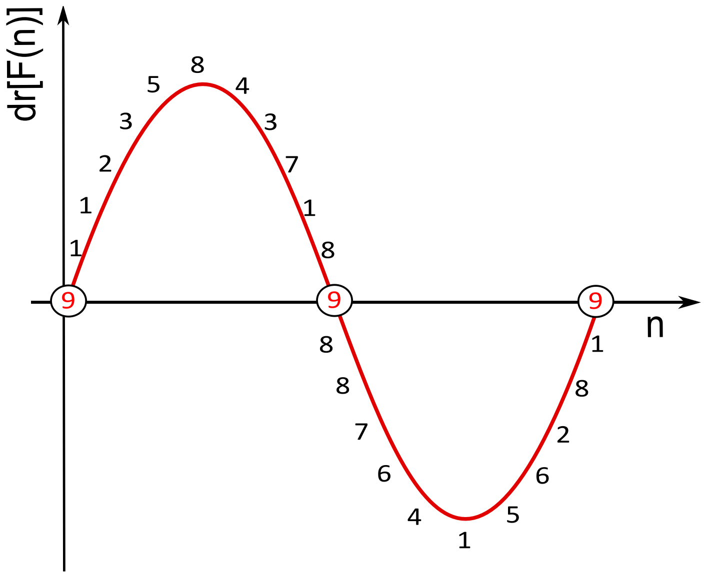

Figure 10. *Distribution of digital roots of individual members of the Fibonacci sequence*

Why the proposed sequence is called the reverse Fibonacci sequence? The proposed sequence is characterized by the same properties as the Fibonacci sequence for the individual values of the digital roots of the members of its sequence, where the values of the digital root of the proposed sequence are reciprocal *(reverse)* to the values of digital roots of the Fibonacci sequence. This reciprocity can be used to write the following relation (37). From this point of view it follows that, like in the Fibonacci sequence, the number **9** can always be obtained from the digital roots of the members of the proposed sequence ***dr[J(n) + J(n+12)].***

$$dr[F(n)] = dr[J(24-n)], for\ n \leq 24 \tag{37}$$

Mutual reversibility of the proposed sequence and Fibonacci sequence is shown in **Figure 11**.

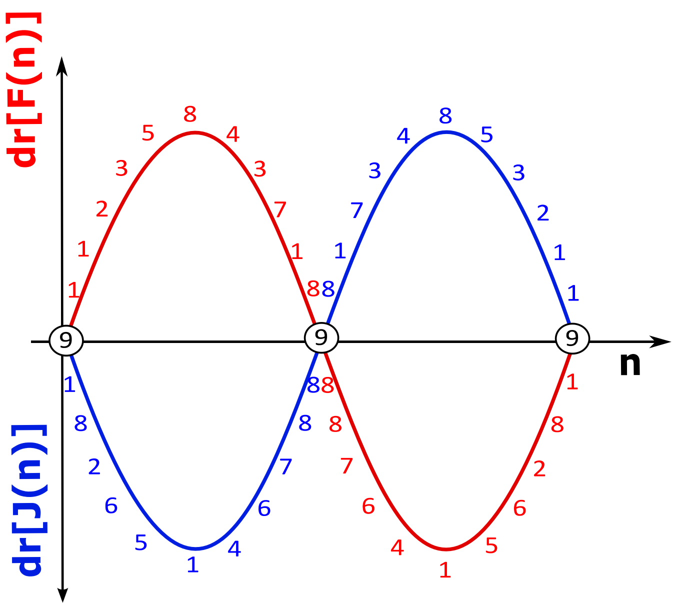

Figure 11. *Distribution of digital roots of individual members of the Fibonacci sequence*

To better illustrate the above mentioned, **Table 1**, in which the individual values of the members of both the proposed and the Fibonacci sequence and their digital roots are quantified, is shown.

From **Table 1** results another property in relation to the Fibonacci sequence. For members of the proposed reverse sequence ***J(19)*** to ***J(2),*** the first digits (*shown in* ***blue***) are identical to the sequence of the first six Fibonacci sequences ***F(1)*** to ***F(6)***. The repetition of this sequence after **6 numbers** is with a **frequency of 3**.

Table 1. *Quantification of individual values of the members of the sequences and their digital roots*

n	*J(n)*	*dr[J(n)]*	*(24-n)*	*F(24-n)*	*dr[F(24-n)]*
0	**0**	*9*	*24*	**46368**	**9**
1	**1**	*1*	*23*	**28657**	**1**
2	**8**	*8*	*22*	**17711**	**8**
3	**56**	*2*	*21*	**10946**	**2**
4	**384**	*6*	*20*	**6765**	**6**
5	**2624**	*5*	*19*	**4181**	**5**
6	**17920**	*1*	*18*	**2584**	**1**
7	**122368**	*4*	*17*	**1597**	**4**
8	**835584**	*6*	*16*	**987**	**6**
9	**5705728**	*7*	*15*	**610**	**7**
10	**38961152**	*8*	*14*	**377**	**8**
11	**266043392**	*8*	*13*	**233**	**8**
12	***1816657920***	*9*	*12*	***144***	*9*
13	**12404916224**	*8*	*11*	**89**	**8**
14	**84706066432**	*1*	*10*	**55**	**1**
15	**578409201664**	*7*	*9*	**34**	**7**
16	**3949625081856**	*3*	*8*	**21**	**3**
17	**26969727041536**	*4*	*7*	**13**	**4**
18	**184160815677440**	*8*	*6*	**8**	**8**
19	**1257528709087232**	*5*	*5*	**5**	**5**
20	**8586943147278336**	*3*	*4*	**3**	**3**
21	**58635315505528832**	*2*	*3*	**2**	**2**
22	**400386978866003968**	*1*	*2*	**1**	**1**
23	**2734013306883801088**	*1*	*1*	**1**	**1**
24	**18669010624142376960**	*9*	*0*	**0**	**9**

Another interesting feature is the periodicity of **digits at the position of ones** (*last digits*) for members of the proposed sequence from member ***J(2)*** with the period **n=24**. The mentioned periodicity is shown in **Figure 12**. It can be seen from the figure below that if a square with vertexes of unit digits equal to **0** is constructed, the side of this square is just equal to the value of $\sqrt{2}$ assuming a unit circle, where $\sqrt{2}$ can be written using equation (17). A similar feature found in 1774 in Fibonacci sequence by a French mathematician **Joseph Louis Lagrange**, who discovered that the digits on the position of ones in the Fibonacci sequence were repeated with a period of **60 places**. [8]

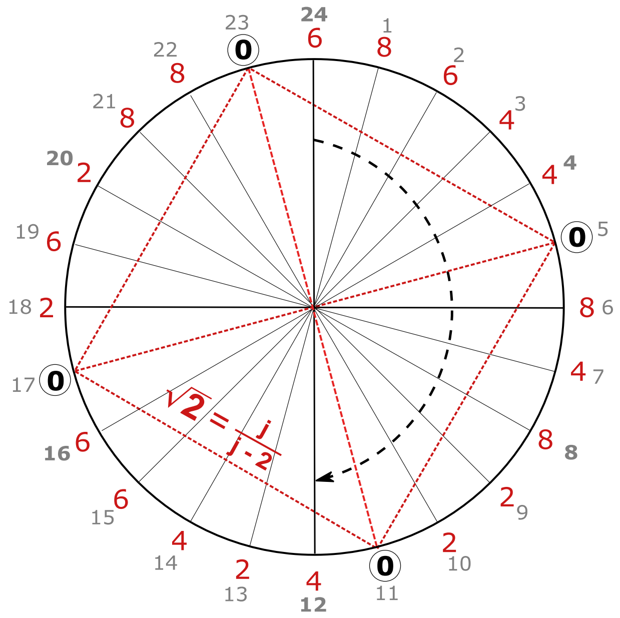

Figure 12. *Periodicity of digits at the position ones for the proposed sequence*

The relation of the proposed sequence to the Fibonacci sequence can also be expressed graphically, see **Figure 13**.

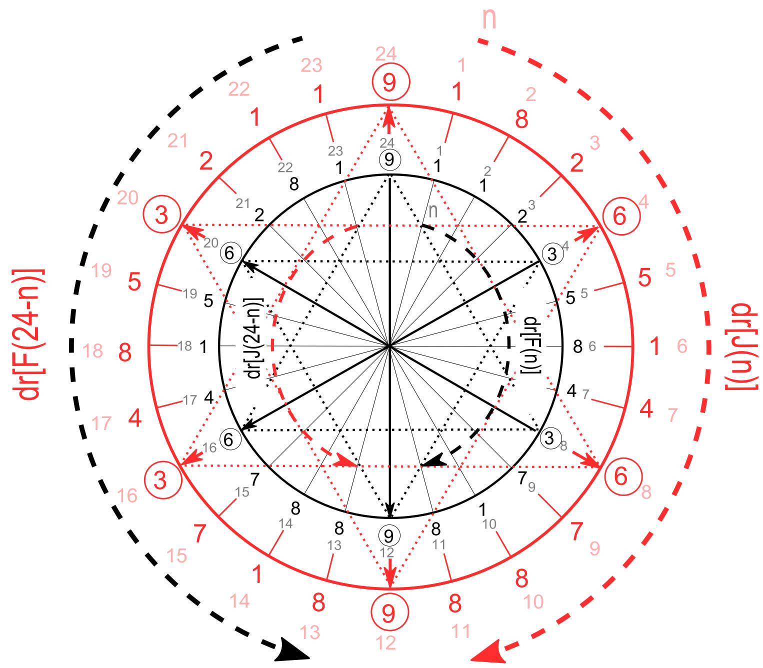

Figure 13. *Graphic representation of relation of Fibonacci sequence and proposed reverse sequence*

Figure 13 shows 2 circles that are divided into **24** equal segments. The first of the circles is **black** and displays the values of the individual members of the *Fibonacci sequence in the clockwise. In the counterclockwise* then are the values of the digital roots of individual members of the proposed *reverse Fibonacci sequence*.

The second circle is shown in **red colour**, where the circle is **just reverse** to the first circle. I.e. it displays the values of the digital roots of the individual members of the proposed *reverse Fibonacci sequence in the clockwise* and *counterclockwise* then the values of the digital roots of individual members of the *Fibonacci sequence*.

For idea, this relationship can be represented as an analog clock where the ticking clockwise represents the Fibonacci sequence and the ticking counterclockwise represents the proposed reverse Fibonacci sequence.

Finally, the above mentioned and other properties of the relation between the Fibonacci sequence and the proposed sequence are described in the following equations:

$$dr[F(n)] = dr[J(24-n)];\ dr[J(n)] = dr[F(24-n)], pro\ n \leq 24, \tag{38}$$

$$dr[F(n) + F(n+12)] = dr[J(n) + J(n+12)] = 9, \tag{39}$$

$$\begin{gathered} dr[J(n) + F(12-n)] = dr[F(n) + J(12-n)] = 9, pro\ n \leq 12, \\ dr[J(n) + F(36-n)] = dr[F(n) + J(36-n)] = 9, pro\ 24 \geq n > 12. \end{gathered} \tag{40}$$

5. Conclusion

The presented article defines a new sequence and a new ratio of the so-called reverse Fibonacci sequence and their relations and properties to other mathematical fundamental numbers and structures, which points to the interest of this sequence and the derived ratio. A definite relation between the original Fibonacci sequence and the proposed reverse Fibonacci sequence and their ratios is proved. The Fibonacci sequence and the golden ratio are applied in nature; therefore, we can assume that in the future the application for the reverse Fibonacci sequence will be discovered in nature. The reverse Fibonacci sequence is hiden in the Fibonacci sequence, therefore all the newly derived properties of the reverse Fibonacci sequence can be used in further golden ratio research and a self-directed field dedicated to the Fibonacci sequence. Also very important is the derived relationship between the golden ratio and the ratio of the reverse Fibonacci sequence. Another important relationship is between the ratio of the reverse Fibonacci sequence and Euler's number ***e***. This article expands our knowledge of the golden ratio and the Fibonacci sequence, and therefore gives hope for widespread use in various disciplines, art and nature.

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